BASIC TENETS OF THE INSTRUCTIONAL APPROACH

Mathematics as Reasoning

Fundamental to the approach taken in this book is my belief that mathematics is first and foremost a form of reasoning, not the performance of endless sequences of procedures invented by others. We do mathematics in our mind, not with our hands or with tools (although *thinking* about what we do with tools can aid tremendously in mathematical thought). To do mathematics is to think in a logical manner; to formulate and test conjectures; to form conclusions, judgments, and inferences. We do mathematics when we recognize patterns, form and manipulate concepts, build and test arguments, invent procedures to solve classes of problems, and decide when to apply procedures we have learned. We do mathematics when we solve problems that have genuine meaning for us. In doing mathematics, we purposefully create, inspect, and manipulate ideas and images to solve problems dealing with quantitative and spatial situations. We reflect on what we know and reorganize it; we make sense of things; we meaningfully and purposefully manipulate mathematical symbols to support our mathematical thought.

Geometry as the Study of Structure

Geometry is the study of ways of organizing or *structuring* our spatial environment and investigating the nature and consequences of that structuring. When we structure something, we determine its nature, shape, or organization by establishing interrelationships between its parts. We structure the plane and space when we organize them by coordinate systems. We structure our visual world when we view it in terms of shapes such as lines, angles, polygons, polyhedra, and geometric transformations.

In this book, students investigate two of the primary classes of shapes used to structure our spatial environment—quadrilaterals and triangles. They examine not only these shapes and relationships between them, but parts of the shapes (such as angles and sides) and interrelationships between these parts. They also develop and refine their geometric and spatial reasoning and problem-solving skills.

Learning Mathematics in a Culture of Inquiry and Sense Making

> *Real comprehension of a notion or a theory implies the re-invention of this theory by the subject. Each time one prematurely teaches a child something he could have discovered himself the child is kept from inventing it and consequently from understanding it completely. Naturally, this does not mean that the teacher has no role anymore, but that his role is less that of a person who gives "lessons" and is rather that of someone who organizes situations that will give rise to curiosity and solution-seeking in the child, and who will support such behavior by means of appropriate arrangements. (Piaget)*

Learning

Students do not learn by receiving or absorbing ready-made ideas from objects or people. Instead, they learn as they reflect on and abstract the mental and physical actions they perform while purposefully interacting with their physical and social environments. As they interact with these environments, they construct mental structures that enable them to make sense of and manage their physical, social, and intellectual experiences.

Like scientists, students are theory builders. They learn as they reorganize their theories, as they discover and adopt more sophisticated and general theories. Such reorganization is triggered by perturbations—that is, by students' realization that their current way of interpreting things does not work or produces unexpected results.

Teaching

The goal of instruction should be to help *each* student build mathematical ideas and theories that are more complex, abstract, and powerful than those he or she currently possesses. The major instructional mechanism for encouraging students' construction of knowledge is the presentation of properly chosen problematic tasks. These tasks guide the direction of students' theory building by properly focusing their attention, encouraging them to reflect on their actions and thoughts, and promoting perturbations that require reorganization of current theories.

However, to be effective, instructional tasks must fall within the students' current *zones of construction*. That is, students' construction of the new concepts required to complete the tasks must be possible given their current conceptual structures and operations. In fact, because students' existing structures determine how they think about all new tasks, as teachers, we must constantly monitor the development of these structures and adjust our instruction accordingly.

Whenever we ignore students' current ways of thinking and attempt to impose methods on students, the sense-making activity of students is stifled. Students mindlessly mimic the methods they are shown. Their belief about the nature of mathematics changes from seeing mathematics as sense-making to seeing mathematics as the learning of set procedures that make little sense. Students change from intellectually autonomous thinkers to teacher/textbook-dependent rule followers.

Establishing a Culture of Inquiry

A fundamental tenet in current research-based, scientific theories of learning mathematics is that instruction should be inquiry-based, with students learning mathematics as they solve problems and share their ideas with one another. To foster meaningful learning in the classroom, teachers, in collaboration with their students, must establish a culture of inquiry in which individuals pose questions, solve problems, share ideas, and think critically. Within this culture, students are involved not only in inquiry, problem solving, and invention, but also in classroom discourse that establishes ideas and truths collaboratively. Students' participation in such a culture promotes their personal construction of ideas as they

- attempt to elaborate and clarify personally developed ideas so that they can communicate them to others;
- reflect on, evaluate, and justify their personally developed ideas in response to challenges posed by classmates;
- attempt to make sense of and sometimes utilize new ideas offered by classmates.

Responsibilities in a Culture of Inquiry

The major responsibilities of teachers and students in a classroom culture of inquiry are as follows.

Students are responsible for

1. Attempting to solve and make sense of all problems given to them.
2. Explaining their mathematical thinking to other members of the class and justifying problem solutions in response to challenges.
3. Listening to, as well as attempting to make sense of, other students' mathematical explanations and problem solutions. This includes
   - asking for clarification if an explanation is not understood;
   - challenging strategies and problem solutions that do not seem reasonable.
4. Working collaboratively with other students. This includes attempting to reach consensus on problem solutions while respecting the rights of others to derive or justify solutions differently.

Teachers are responsible for

1. Selecting instructional tasks and guiding students' work on these tasks so that students' thinking becomes increasingly more sophisticated. This includes
   - choosing sequences of problematic tasks that are based on detailed knowledge of how students construct meanings for specific mathematical topics as well as the conceptual advances that students can make with those topics during the course of instruction;
   - continually assessing students' learning progress and adjusting instruction accordingly;
   - encouraging students to reflect on their mathematical experiences.

2. Establishing a social environment that supports a spirit of inquiry and collaborative small-group work. This includes
   - explaining;
   - illustrating with classroom examples;
   - regularly reminding students of their responsibilities (as described earlier).
3. Encouraging productive dialogue among students. This includes
   - encouraging students to explain and justify their mathematical ideas;
   - highlighting conflicts between alternative student interpretations or solutions;
   - unobtrusively encouraging potentially fruitful student contributions;
   - redescribing students' ideas in more sophisticated ways that students can still comprehend;
   - introducing mathematical concepts, symbolism, and terminology at appropriate times so that students can use them to reflect on and communicate about their own developing ideas.

Helping Students Meet Their Responsibilities. If students are not accustomed to participating in a culture of inquiry in mathematics class, it may take them several weeks to become comfortable with and competent within such an instructional environment. They will need regular and explicit discussion of this new way of learning. Posting student responsibilities on a poster board in class and regularly asking students what their responsibilities are and how they are implementing them can help.

Students will also need to see classroom examples of their responsibilities in action. For instance, to learn how to explain and justify their strategies and problem solutions, students need to talk about the processes of explanation and justification. If you see that a number of students are having difficulty explaining their thinking about a particular problem, make such explanations the focus of a class discussion. You might start the discussion by asking students who are having difficulty articulating an idea to explain it as best they can to the class. Then ask other students who used a similar strategy how they explained it: "How do you think we should talk about these ideas? What words should we use to refer to what you are talking about?" Such discussions can help students develop a language and set of conceptualizations for describing their developing ideas.

Understanding Students' Geometric Thinking

A considerable amount of research has established the van Hiele theory as an accurate description of the development of students' geometric thinking. Knowledge of these levels is essential in designing, conducting, and evaluating meaningful geometry instruction.

The Van Hiele Levels

According to van Hiele, students progress through several levels of qualitatively different and increasingly sophisticated levels of thought in geometry.[1] They pass through these levels sequentially. Consequently, students who are required by instruction to study content at a higher level than they have achieved cannot make sense of that content, and they resort to memorization. Furthermore, people who reason at different levels may use the same terms but have very different meanings for those terms. Thus, effective communication between people at different levels—especially teacher and students—can be difficult.

In the van Hiele theory, a critical factor used in distinguishing levels of thinking is how students deal with *geometric properties*. Such properties describe spatial relationships between parts of shapes. For example, these statements describe geometric properties:

a. Opposite sides of this quadrilateral are congruent.
b. Opposite sides of parallelograms are parallel.
c. A rectangle has four right angles.
d. Adjacent angles of parallelograms are supplementary.
e. This quadrilateral has one line of symmetry.

[1]For more details see D. H. Clements and M. T. Battista, "Geometry and Spatial Reasoning" in *Handbook of Research on Mathematics Teaching and Learning*, ed. D. Grouws (New York: NCTM/Macmillan, 1992), 420–464.

Examples (a) and (b) explicitly describe relationships between the sides, or parts, of quadrilaterals. By specifying the measures of angles, examples (c) and (d) describe relationships between pairs of adjacent sides. Finally, example (e) describes a relationship between two "halves" of a quadrilateral, which, with further analysis, could be described in terms of line segments and angles.

Level 1: Visual. At the first van Hiele level, students identify and reason about shapes and other geometric configurations according to their appearance. Their thinking is dominated by perception. They recognize and mentally represent shapes such as squares and triangles as visual wholes. When identifying shapes, students often use visual prototypes, saying that a given figure is a rectangle, for instance, because "it looks like a door." Students at the visual level do not attend to geometric properties of shapes. For example, they might distinguish one shape from another without referring to a single property of either shape. Instead, they might judge that two shapes are congruent because they look the same or because they can be turned to look the same.

Level 2: Descriptive/Analytic. At the second van Hiele level, students recognize and can characterize shapes by their properties, that is, by spatial relationships between their parts. For instance, students might think of a rectangle as a figure that has opposite sides equal and parallel as well as having four right angles. Though still important, the appearance of shapes becomes secondary because students conceptualize shapes as being determined by collections of properties rather than as simply matching visual prototypes. Properties are established experimentally by observing, measuring, drawing, and model making. However, students tend to name all the properties they know for a class of shapes, rather than a sufficient set. They also do not see relationships between classes of shapes (e.g., a student might contend that a figure is not a rectangle because it is a square).

Of course, which properties students attribute to shapes depends on their experiences with those shapes. For instance, although students typically come to see that rectangles have right angles and congruent and parallel opposite sides, it is not likely that they will notice that the diagonals bisect each other unless they have had sufficient experience analyzing diagonals. Furthermore, some students formulate incorrect properties of shapes. For example, many middle-school students think that rectangles cannot have all sides congruent.

Level 3: Abstract/Relational. At the third van Hiele level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a class of shapes, and understand and sometimes even provide logical arguments in the geometric domain. They can classify shapes hierarchically and give informal arguments to justify their classifications (e.g., a square is identified as a rhombus because it can be thought of as a "rhombus with some extra properties"). They can understand why, and are willing to accept that, a square is a rectangle. They can discover properties of classes of figures by informal deduction. For example, they might deduce that in any quadrilateral the sum of the angles must be 360° because any quadrilateral can be divided into two triangles, each of whose angles sum to 180°. However, for students at this level, "any trapezoid" may actually mean "all the trapezoids with which I am familiar," not necessarily all possible trapezoids.

Because students see that some properties imply others, they no longer feel a need to list all the properties of a class of shapes. Definitions are seen not merely as descriptions of shapes but as a way of logically organizing properties. The students still, however, do not grasp that logical deduction is the method for establishing geometric truths.

Level 4: Formal Deduction and Proof. At the fourth van Hiele level, students can formally prove theorems within an axiomatic system. That is, they can produce a sequence of statements that logically justifies a conclusion as a consequence of the "givens." They recognize the difference among undefined terms, definitions, axioms, and theorems. They can reason by employing formal logic to interpret geometric statements.

Thinking at level 4 is required for a proof-oriented high school geometry course.

Of course, an even higher level of thought is needed to analyze and compare different axiomatic systems.

Transitions Between Levels. During the transition from the visual to the descriptive/analytic level, students start attending to the components of shapes such as sides and angles. They start examining the relationships between these components. But often their descriptions of these relationships are intuitive, visual, and imprecise. For instance, students might say that the difference between a rectangle and a parallelogram with no right angles is that the sides of the parallelogram are tilted and the sides of the rectangle are straight.

In the transition from the descriptive/analytic to the abstract/relational level, students begin to discover that some combinations of properties of a class of shapes imply other properties. For instance, they might claim that because a rectangle has opposite sides parallel, the opposite sides must be equal.

At What Levels Are Middle-Grade Students Functioning?

Most middle-grade and junior-high students are functioning at van Hiele level 1 or 2.[2] In fact, more than 70% of students begin high school geometry at level 1 or below. Unfortunately, research indicates that only students who enter at level 2 or higher have a good chance of becoming competent with proof—a level 4 activity—by the end of the course. Because so many students are functioning at such low van Hiele levels when they enter high school geometry, such courses are dismally ineffective. Indeed, almost 40% of students end the year below level 2, and only about 30% in courses that teach proof reach a 75% mastery level in proof writing.

Thus, a major goal for middle-grade and junior-high geometry curricula—and that of the *Shape Makers* curriculum—should be to help students move from level 1 to level 2, then to level 3 in the van Hiele hierarchy. Students who have worked through the *Shape Makers* curriculum will be well prepared for high school geometry courses that start with further development and expansion of students' level 3 reasoning before dealing with formal proof.

[2] See Clements and Battista 1992, for more details on this research.

THE *SHAPE MAKERS* COMPUTER MICROWORLD

The usual approach to teaching students about classes of geometric shapes is to define the shapes. However, as was discussed earlier, only students at van Hiele level 3 can fully understand definitions such as "A rectangle is a quadrilateral with four right angles" or "A square is a rectangle with all sides equal." So the definitional approach is too formal to have much chance of success with most students. It forces students to attempt to find their way in what, for them, is an incomprehensible maze of meaningless abstractions. It is little wonder that so few students succeed, and that fewer still enjoy what they are doing.

Shape Makers provides an alternative approach. Students can manipulate the Shape Makers just as they can manipulate a physical apparatus. Through their actions and reflection on those actions, students can discover properties of the Shape Makers that coincide with those of the classes of shapes made by the Shape Makers. In essence, students can learn about properties and classes of shapes using the same processes they use in learning everyday concepts such as "chair" or "book." That is, they can manipulate and reflect on numerous examples instead of trying to comprehend verbal definitions. Eventually, after extensive visual investigations have enabled students to understand shapes in terms of their properties, students can deal meaningfully with geometric definitions.

Dynamic Mental Models

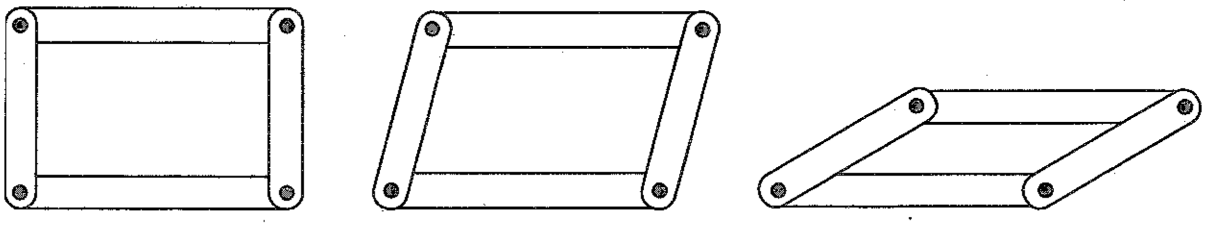
The *Shape Makers* microworld is designed to promote the development of dynamic mental models for thinking about geometric shapes and their manipulation. Mental models are mentally constructed representations of real-world situations. They are derived from our experiences and our reflections on those experiences, and they usually have an image-like quality. Reasoning with mental models is like reasoning about physical objects. When using a mental model to reason about a situation, a person can mentally move around, move on or into, combine, and transform objects, as well as perform other operations like those that can be performed on objects in the physical world. Students draw inferences by mentally manipulating mental models and observing the results.

The primary source of mental models is our experience in dealing with the world, especially with physical objects. To think of how a mental model for a parallelogram might be derived from real-world manipulation, imagine four straight rods connected at their endpoints in a way that permits freedom of movement at the connections—a movable quadrilateral. Imagine now that the opposite rods are the same length. No matter how we move this physical apparatus, it always forms a parallelogram, and sometimes a rectangle (see Figure 1).

As we manipulate our "parallelogram maker," we not only see how its shape changes, we feel the physical constraints that we have built into it. We see and feel how one parallelogram is related to others. The visual and kinesthetic experiences that we abstract from our actions with this apparatus, along with our reflections on those actions, are integrated to form a mental model for a parallelogram, a model that we can use in reasoning about parallelograms.

The power of utilizing mental models to reason about geometry can be illustrated by the case of a second-grader who had been contemplating the notion that squares are special types of rectangles.

Figure 1.



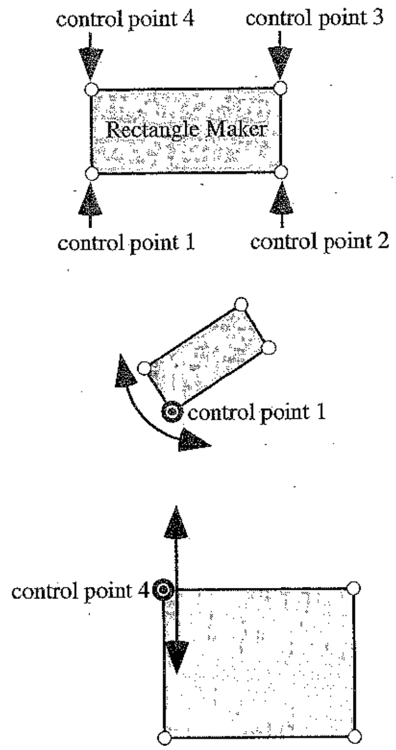
She made sense of the idea not by referring to verbally stated properties of these shapes, but by thinking about how some "stretchy square bathroom things" could be stretched into rectangles.[3] She reasoned by performing a simulation of changing a square into a rectangle using the mental model she had derived from her physical actions with the "stretchy square bathroom things." Furthermore, she performed a special type of visual transformation, one that incorporated some formal mathematical constraints—preserving 90° angles—into her mental model. Her reasoning was intuitively constrained by her emerging knowledge of the properties of shapes. Such reasoning is the key to meaningful and powerful geometric thinking.

The *Shape Makers* Computer Microworld

The *Shape Makers* computer microworld is built upon The Geometer's Sketchpad®, a software tool for constructing and investigating geometry dynamically. The *Shape Makers* microworld is designed to help students construct appropriate mental models for thinking about various types of quadrilaterals and triangles. In this microworld, each class of common quadrilaterals and triangles has a Shape Maker, a Sketchpad™ construction that can be dynamically transformed in various ways, but only to produce different shapes in the same class. Not only can these Shape Makers be manipulated like the physical parallelogram maker described above, but their side lengths can also be changed. The Rectangle Maker, for instance, can be manipulated to make any desired rectangle that fits on the computer screen, no matter what its shape, size, or orientation—but it can make only rectangles (see Figure 2).

The Rectangle Maker's shape is changed by using the mouse to drag one of its control points. A *control point* is represented by a small circle at one of the vertices of the Rectangle Maker. To drag a control point from one location to another, point to it with the arrow, press and

Figure 2



hold down the mouse button (the point becomes highlighted, indicating that it is selected), then move the mouse, keeping the button held down. Release the mouse button when the control point is at the desired location.

The Rectangle Maker can be scaled or turned with control point 1. You can make it taller or shorter by dragging control point 4. (Although the exact functions of the different control points vary among Shape Makers, you can always change a Shape Maker's size and position, along with all of its critical attributes, such as side lengths and angle measures.)

You can move the Rectangle Maker from one screen location to another by dragging its interior. When the Rectangle Maker is in the desired position, click somewhere off the Rectangle Maker to deselect it.

There are quadrilateral Shape Makers for squares, rectangles, parallelograms, kites, rhombuses, trapezoids, and general quadrilaterals. There are triangle Shape Makers for general, isosceles, equilateral, and right triangles.

[3] M. T. Battista, "On Greeno's Environmental/Model View of Conceptual Domains: A Spatial/Geometric Perspective," *Journal for Research in Mathematics Education* 25(January 1994): 86–94.

It is important to recognize that the *Shape Makers* computer microworld has been constructed in The Geometer's Sketchpad, which is a comprehensive, dynamic geometry construction program. In Sketchpad, students can construct, measure, and transform geometric shapes on the computer screen. Measurements are instantaneously updated as geometric objects are altered. For instance, if you construct and measure the line segment between two points, then move one of the points, the lengths of the newly created segments are displayed continuously. Furthermore, geometric constraints that are built into constructions can be preserved as the shapes are varied. For instance, if you construct a polygon and reflect it about a line, then move one of the polygon's vertices, the resulting changes to the reflected image are made automatically and instantaneously.

Students who are working with *Shape Makers* need to know very little about using Sketchpad—what they do need to know is explained within the activities. (In fact, so that students don't inadvertently activate tools they don't need in the activities, you may want to direct students to hide the toolbox by clicking its close box.) The only tool students need is the **Selection Arrow** tool, simply called the "arrow" in this book. However, there are times when you or your students might enjoy extending *Shape Maker* analyses and activities using commands in Sketchpad. To become familiar with the possibilities, see *The Geometer's Sketchpad Learning Guide* and *Reference Manual.* In fact, one of the advantages of using *Shape Makers* is that students start to become familiar with a computer tool—Sketchpad—that they can productively use for the rest of their mathematical careers.

The *Shape Makers* CD

The CD that accompanies this book is a hybrid disc that will work both on Macintosh® computers and computers running Microsoft® Windows 95 or NT 4.0 or later. It contains Sketchpad Shape Maker sketches for both operating systems, but when you insert the CD in your computer, you will see only the sketches for that computer's operating system. You can run the sketches from the CD, or you may copy them to as many hard disks in your classroom or lab as you like. You may also copy them to a network file server.

Accessing Shape Maker Sketches

Shape Makers are contained in special Sketchpad files called sketches. Once the Sketchpad program and Shape Maker sketches have been loaded onto a computer's hard disk, and Sketchpad has been activated, Shape Maker sketches can be accessed by selecting **Open** from the Sketchpad File menu. In the Open dialog box that then appears, open the appropriate folder by clicking the triangle (▷) to make it downward pointing (▽) so that the Shape Maker sketches are visible, then select the Shape Maker sketch or sketches for that particular activity by double-clicking them. The organization and location of the sketches on the Shape Makers disk are shown in the File Organization Table (Figure 3) on the following page.

In the description of the instructional activities, the software needed for each exploration is listed before the Required Materials chart. For example, the two Shape Maker sketches needed for Quadrilateral Exploration 1 are displayed as shown in Figure 4 on the following page.

On student sheets, required sketches are listed the same way, right after the title of the activity.

Important Note. Any time the computer asks students whether they want to save their work, they must click Don't Save. Otherwise, the changes will be made to the original Shape Maker files on the hard disk. If you want students to save their work, they must use the Save As command.