"The van Hiele Model of the Development of Geometric Thought"

by

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The van Hiele Model of the Development of Geometric Thought

Mary L. Crowley

HAVE you ever had students who could recognize a square but not define it? Have you ever noticed that some students do not understand that a square is a rectangle? Have you ever had students who complain about having to prove something they already "know"? According to two Dutch educators, Dina van Hiele-Geldof and her husband, Pierre Marie van Hiele, behaviors such as these reflect a student's level of geometric maturity. Have you ever wondered how to help your students achieve a more sophisticated level of geometric thinking? The van Hiele model of geometric thought can be used to guide instruction as well as assess student abilities. This article presents an overview of the model and discusses its classroom implications.

The van Hiele model of geometric thought emerged from the doctoral works of Dina van Hiele-Geldof (1984a) and Pierre van Hiele (1984b), which were completed simultaneously at the University of Utrecht. Since Dina died shortly after finishing her dissertation, it was Pierre who clarified, amended, and advanced the theory. With the exception of the Soviet Union, where the geometry curriculum was revised in the 1960s to conform to the van Hiele model, the work was slow in gaining international attention. It was not until the 1970s that a North American, Izaak Wirszup (1976), began to write and speak about the model. At about the same time, Hans Freudenthal, the van Hieles' professor from Utrecht, called attention to their works in his titanic book, *Mathematics as an Educational Task* (1973). During the past decade there has been increased North American interest in the van Hieles' contributions. This has been particularly enhanced by the 1984 translations into English of some of the major works of the couple (Geddes, Fuys, and Tischler 1984).

The model consists of five levels of understanding. The levels, labeled "visualization," "analysis," "informal deduction," "formal deduction," and "rigor" (Shaughnessy and Burger 1985, p. 420) describe characteristics of the thinking process. Assisted by appropriate instructional experiences, the model asserts that the learner moves sequentially from the initial, or basic, level (visualization), where space is simply observed—the properties of figures are not explicitly recognized, through the sequence listed above to the highest level (rigor), which is concerned with formal abstract aspects of deduction. Few students are exposed to, or reach, the latter level. A synopsis of the levels is presented below.

The Model

Level 0[1] (Basic Level): Visualization

At this initial stage, students are aware of space only as something that exists around them. Geometric concepts are viewed as total entities rather than as having components or attributes. Geometric figures, for example, are recognized by their shape as a whole, that is, by their physical appearance, not by their parts or properties. A person functioning at this level can learn geometric vocabulary, can identify specified shapes, and given a figure, can reproduce it. For example, given the diagrams in figure 1.1, a student at this level would be able to recognize that there are squares in (a) and rectangles in (b) because these are similar in shape to previously encountered squares and rectangles. Furthermore, given a geoboard or paper, the student could copy the shapes. A person at this stage, however, would not recognize that the figures have right angles or that opposite sides are parallel.

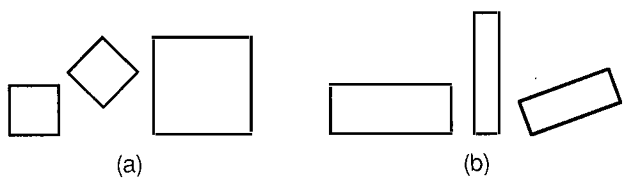


Fig. 1.1

Level 1: Analysis

At level 1, an analysis of geometric concepts begins. For example, through observation and experimentation students begin to discern the characteristics of figures. These emerging properties are then used to conceptualize classes of shapes. Thus figures are recognized as having parts and are recognized by their parts. Given a grid of parallelograms such as those in figure 1.2, students could, by "coloring" the equal angles, "establish" that the opposite angles of parallelograms are equal. After using several such examples, students could make generalizations for the class of parallelograms. Relationships between properties, however, cannot yet be explained by students at this level, interrelationships between figures are still not seen, and definitions are not yet understood.

1. Different numbering systems for the model may be encountered in the literature. The van Hieles themselves spoke of levels beginning with the basic level, or level 0, and ending with level 4.

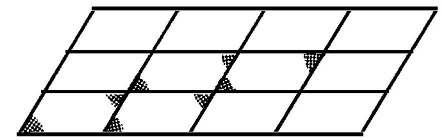


Fig. 1.2

Level 2: Informal Deduction

At this level, students can establish the interrelationships of properties both within figures (e.g., in a quadrilateral, opposite sides being parallel necessitates opposite angles being equal) and among figures (a square is a rectangle because it has all the properties of a rectangle). Thus they can deduce properties of a figure and recognize classes of figures. Class inclusion is understood. Definitions are meaningful. Informal arguments can be followed and given. The student at this level, however, does not comprehend the significance of deduction as a whole or the role of axioms. Empirically obtained results are often used in conjunction with deduction techniques. Formal proofs can be followed, but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.

Level 3: Deduction

At this level, the significance of deduction as a way of establishing geometric theory within an axiomatic system is understood. The interrelationship and role of undefined terms, axioms, postulates, definitions, theorems, and proof is seen. A person at this level can construct, not just memorize, proofs; the possibility of developing a proof in more than one way is seen; the interaction of necessary and sufficient conditions is understood; distinctions between a statement and its converse can be made.

Level 4: Rigor

At this stage the learner can work in a variety of axiomatic systems, that is, non-Euclidean geometries can be studied, and different systems can be compared. Geometry is seen in the abstract.

This last level is the least developed in the original works and has received little attention from researchers. P. M. van Hiele has acknowledged that he is interested in the first three levels in particular (Alan Hoffer, personal communication, 25 February 1985). Since the majority of high school geometry courses are taught at level 3, it is not surprising that most research has also concentrated on lower levels. Perhaps as the van Hiele model is extended to other areas (it is being applied to economics and chemistry in Holland), this last level will achieve more prominence.

Properties of the Model

In addition to furnishing insights into the thinking that is specific to each level of geometric thought, the van Hieles identified some generalities that characterize the model. These properties are particularly significant to educators because they provide guidance for making instructional decisions.

1. *Sequential.* As with most developmental theories, a person must proceed through the levels in order. To function successfully at a particular level, a learner must have acquired the strategies of the preceding levels.

2. *Advancement.* Progress (or lack of it) from level to level depends more on the content and methods of instruction received than on age: No method of instruction allows a student to skip a level; some methods enhance progress, whereas others retard or even prevent movement between levels. Van Hiele points out that it is possible to teach "a skillful pupil abilities above his actual level, like one can train young children in the arithmetic of fractions without telling them what fractions mean, or older children in differentiating and integrating though they do no know what differential quotients and integrals are" (Freudenthal 1973, p. 25). Geometric examples include the memorization of an area formula or relationships like "a square is a rectangle." In situations like these, what has actually happened is that the subject matter has been reduced to a lower level and understanding has not occurred.

3. *Intrinsic and extrinsic.* The inherent objects at one level become the objects of study at the next level. For example, at level 0 only the form of a figure is perceived. The figure is, of course, determined by its properties, but it is not until level 1 that the figure is analyzed and its components and properties are discovered.

4. *Linguistics.* "Each level has its own linguistic symbols and its own systems of relations connecting these symbols" (P. van Hiele 1984a, p. 246). Thus a relation that is "correct" at one level may be modified at another level. For example, a figure may have more than one name (class inclusion)—a square is also a rectangle (and a parallelogram!). A student at level 1 does not conceptualize that this kind of nesting can occur. This type of notion and its accompanying language, however, are fundamental at level 2.

5. *Mismatch.* If the student is at one level and instruction is at a different level, the desired learning and progress may not occur. In particular, if the teacher, instructional materials, content, vocabulary, and so on, are at a higher level than the learner, the student will not be able to follow the thought processes being used.

Phases of Learning

As was indicated above, the van Hieles assert that progress through the levels is more dependent on the instruction received than on age or maturation. Thus the method and organization of instruction, as well as the content and materials used, are important areas of pedagogical concern. To address these issues, the van Hieles proposed five sequential phases of learning: inquiry, directed orientation, explication, free orientation, and integration. They assert that instruction developed according to this sequence promotes the acquisition of a level (van Hiele-Geldof 1984b). Sample activities from level-2 work with the rhombus are used here to illustrate.

Phase 1: Inquiry/Information

At this initial stage, the teacher and students engage in conversation and activity about the objects of study for this level. Observations are made, questions are raised, and level-specific vocabulary is introduced (Hoffer 1983, p. 208). For example, the teacher asks students, "What is a rhombus? A square? A parallelogram? How are they alike? Different? Do you think a square could be a rhombus? Could a rhombus be a square? Why do you say that? . . ." The purpose of these activities is twofold: (1) the teacher learns what prior knowledge the students have about the topic, and (2) the students learn what direction further study will take.

Phase 2: Directed Orientation

The students explore the topic of study through materials that the teacher has carefully sequenced. These activities should gradually reveal to the students the structures characteristic of this level. Thus, much of the material will be short tasks designed to elicit specific responses. For example, the teacher might ask students to use a geoboard to construct a rhombus with equal diagonals, to construct another that is larger, to construct another that is smaller. Another activity would be to build a rhombus with four right angles, then three right angles, two right angles, one right angle. . . .

Phase 3: Explication

Building on their previous experiences, students express and exchange their emerging views about the structures that have been observed. Other than to assist students in using accurate and appropriate language, the teacher's role is minimal. It is during this phase that the level's system of relations begins to become apparent. Continuing the rhombus example, students would discuss with each other and the teacher what figures and properties emerged in the activities above.

Phase 4: Free Orientation

The student encounters more complex tasks—tasks with many steps, tasks that can be completed in several ways, and open-ended tasks. "They gain experience in finding their own way or resolving the tasks. By orienting themselves in the field of investigation, many relations between the objects of study become explicit to the students" (Hoffer 1983, p. 208). For example, students would complete an activity such as the following. "Fold a piece of paper in half, then in half again as shown here (fig. 1.3a). Try to imagine what kind of figure you would get if you cut off the corner made by the folds (fig. 1.3b). Justify your answer before you cut. What type(s) of figures do you get if you cut the corner at a 30° angle? At a 45° angle? Describe the angles at the point of intersection of the diagonals. The point of intersection is at what point on the diagonals? Why is the area of a rhombus described by one-half the product of the two diagonals?"

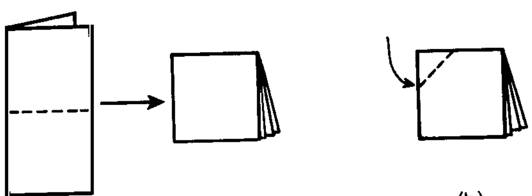


Fig. 1.3

Phase 5: Integration

The students review and summarize what they have learned with the goal of forming an overview of the new network of objects and relations. The teacher can assist in this synthesis "by furnishing global surveys" (van Hiele 1984a, p. 247) of what the students have learned. It is important, however, that these summaries not present anything new. The properties of the rhombus that have emerged would be summarized and their origins reviewed.

At the end of the fifth phase, students have attained a new level of thought. The new domain of thinking replaces the old, and students are ready to repeat the phases of learning at the next level.

Providing van Hiele–Based Experiences

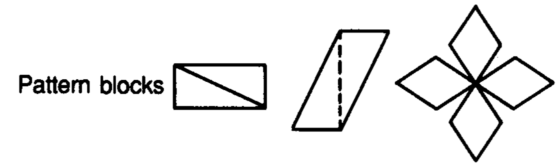
Implicit in the van Hieles' writing is the notion that children should be presented with a wide variety of geometric experiences. Teachers in the early elementary years can provide basic-level exploratory experiences through cutouts, geoboards, paper folding, D-sticks, straws, grid work, tessellations, tangrams, and geometric puzzles. Middle school and junior high school experiences, roughly at levels 1 and 2, can include working with grids, collections of shapes, "property cards," "family trees," and "what's

my name" games. The following pages provide examples of these and other types of activities appropriate for the first four van Hiele levels. Many of these ideas were culled from the descriptors of student behavior developed by the researchers at Brooklyn College (Geddes et al. 1985). Additional activities can be found in the articles by Burger (1985), Burger and Shaughnessy (1986), Hoffer (1981), Prevost (1985), and Shaughnessy and Burger (1985).

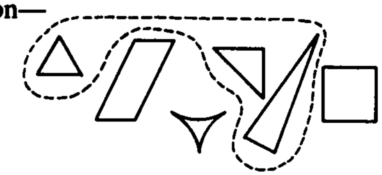
Basic Level (Visualization): Geometric shapes are recognized on the basis of their physical appearance as a whole.

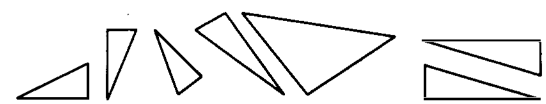
Provide students opportunities—

1. to manipulate, color, fold, and construct geometric shapes

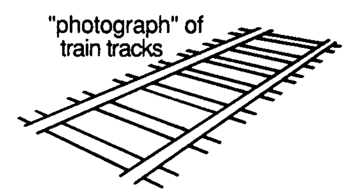


2. to identify a shape or geometric relation—
   - in a simple drawing
   - in a set of cutouts, pattern blocks, or other manipulatives (i.e., sort)
   - in a variety of orientations

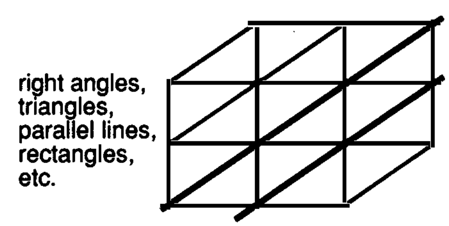




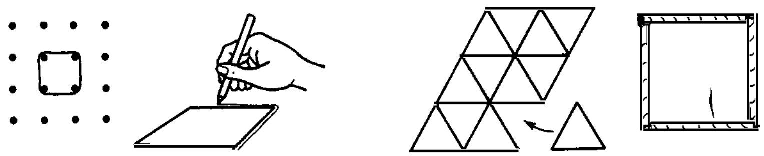
   - involving physical objects in the classroom, the home, photographs, and other places



   - within other shapes



3. to create shapes—
   - by copying figures on dot paper, grid paper, or tracing paper, by using geoboards and circular geoboards, or by tracing cutouts
     by drawing figures
     by constructing figures with sticks, straws, or pipe cleaners or by tiling with manipulatives, pattern blocks, and so on

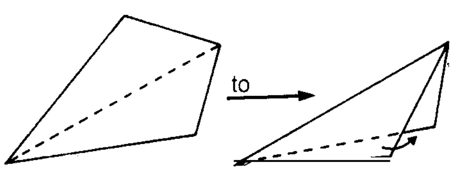


4. to describe geometric shapes and constructs verbally using appropriate standard and nonstandard language
   - a cube "looks like a block or a box"
   - "corners" for angles
5. to work on problems that can be solved by managing shapes, measuring, and counting
   Find the area of a box top by tiling and counting.
   Use two triangular shapes to make a rectangle; another triangle (tangrams).

Level 1 (Analysis): Form recedes and the properties of figures emerge.

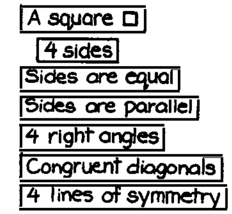
Provide students opportunities—

1. to measure, color, fold, cut, model, and tile in order to identify properties of figures and other geometric relationships
   - fold a kite on a diagonal and examine the "fit."

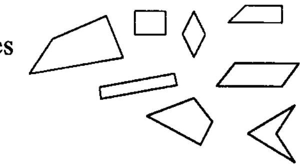


2. to describe a class of figures by its properties (charts, verbally, "property cards")

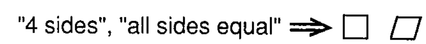
   "Without using a picture, how would you describe a [figure] to someone who has never seen one?"
   - property cards



3. to compare shapes according to their characterizing properties
   - Note how a square and a rhombus are alike, are different in regard to angles, . . . in regard to sides.
4. to sort and resort shapes by single attributes
   - Sort cutouts of quadrilaterals by
     —number of parallel sides
     —number of right angles

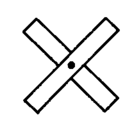


5. to identify and draw a figure given an oral or written description of its properties
   - Teachers or students describe a figure verbally and ask for (all possible) figures with those properties.
   - "What's my name"—reveal clues (properties) one by one, pausing after each, until students can accurately identify the figure. This can be done on an overhead, piece of paper, property cards.



6. to identify a shape from visual clues
   - gradually reveal a shape, asking students to identify at each stage possible names for the shape.

7. to empirically derive (from studying many examples) "rules" and generalizations
   - From tiling and measuring many rectangles, students see that "$b \times h$" is a shortcut for adding the number of tiles.
8. to identify properties that can be used to characterize or contrast different classes of figures
   - Ask, "Opposite sides equal describes . . ."
   - Explore the relationship between diagonals and figures by joining two cardboard strips. A square is generated by the end points when . . . (the diagonals are congruent,



bisect each other, and meet at right angles). Change the angle and the diagonals determine . . . (a rectangle). Noncongruent diagonals generate . . .

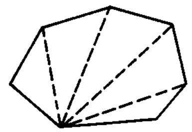
9. to discover properties of unfamiliar classes of objects

   From examples and nonexamples of trapezoids, determine the properties of trapezoids.

10. to encounter and use appropriate vocabulary and symbols

11. to solve geometric problems that require knowing properties of figures, geometric relationships, or insightful approaches

    - Without measuring, find the sum of the angles in a septagon. (Insightful students will "see" triangles, that is, relate this to known figures.)

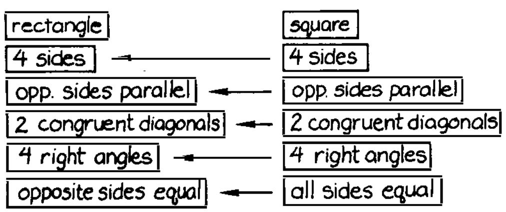


Level 2 (Informal Deduction): A network of relations begins to form.

Provide students opportunities—

1. to study relationships developed at level 1, looking for inclusions and implications

   Use property cards:



   Working on a geoboard, change a quadrilateral to a trapezoid, trapezoid to parallelogram, parallelogram to rectangle. . . . What was required in each transformation?

2. to identify minimum sets of properties that describe a figure

   Students could compete and check each other in this. Ask students how they would describe a figure to someone. Could they use fewer steps? Different steps?

3. to develop and use definitions

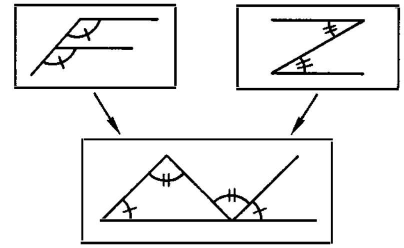
   A square is . . .

4. to follow informal arguments

5. to present informal arguments (using diagrams, cut-out shapes, flow charts)

   Ancestry mappings: Use cards and arrows to display the "origins" or

"family tree" of an idea—for example, "The exterior angle of a triangle equals the sum of the opposite interior angles."



6. to follow deductive arguments, perhaps supplying a few "missing steps"

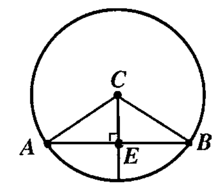
   - *C* is the center of the circle. Why is—

     a) $AC \cong BC$

     b) $\angle CAB \cong \angle CBA$

     c) $\triangle ACE \cong \triangle BCE$

     d) $AE \cong EB$



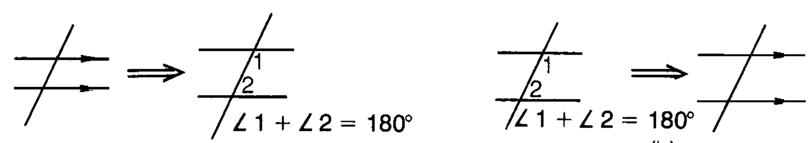
   Note: Reasons other than the level-0 response, "It looks like . . .," must be given for this to be level 2.

7. to attempt to provide more than one approach or explanation

   - Define a parallelogram in two ways (i.e., "4 sides, opposite sides parallel" or "4 sides, opposite sides congruent").

8. to work with and discuss situations that highlight a statement and its converse

   - Write the converse of this statement: If a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are supplementary. Which diagram correctly reflects the converse?

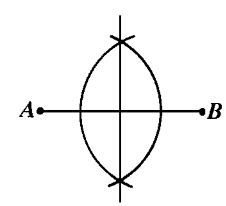


(a) (b)

   - State the converse of the following statement and discuss its validity: "If it's raining, I'm wearing boots."

9. to solve problems where properties of figures and interrelationships are important

   To construct the bisector of a line segment, sweep out two arcs of equal radii (as shown). Explain why the line through the points of intersection of



the arcs is the perpendicular bisector of the segment (i.e., use the properties of a rhombus).

Level 3 (Formal Deduction): The nature of deduction is understood. . . .

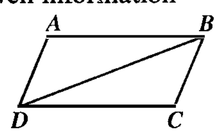
Provide students opportunities—

1. to identify what is given and what is to be proved in a problem

   For the following problem, identify what is known and what is to be proved or shown. Do *NOT* complete the proof. "The perpendicular bisector of the base of an isosceles triangle passes through the vertex of the triangle."

2. to identify information implied by a figure or by given information

   Figure *ABCD* is a parallelogram. Discuss what you know about this figure. Write a problem in "If . . . then . . ." form based on this figure.



3. to demonstrate an understanding of the meaning of *undefined term, postulate, theorem, definition,* etc.,

   Which of the following statements is a postulate, a theorem, a definition? Why?

   a) Points that lie on the same line are called collinear. (D)
   b) Two points determine a line. (P)
   c) Every segment has exactly one midpoint. (T)
   d) The midpoint of a segment is said to bisect the segment. (D)

4. to demonstrate an understanding of necessary and sufficient conditions

   Write a *definition* of a square that begins

   a) A square is a quadrilateral . . .
   b) A square is a parallelogram . . .
   c) A square is a rectangle . . .
   d) A square is a rhombus . . .

5. to prove rigorously the relationships developed informally at level 2
6. to prove unfamiliar relationships
7. to compare different proofs of a theorem—for example, the Pythagorean theorem
8. to use a variety of techniques of proof—for example, synthetic, transformations, coordinates, vectors
9. to identify general strategies of proof
   - If a proof involves parallelism, try "saws," "ladders," or rotations of 180°.

10. to think about geometric thinking
    - The following situations involve deductive or inductive thinking. Identify which type of thinking is involved and why.

      a) All goats have a beard. Sandy is a goat. Thus, Sandy has a beard.
      b) After measuring the angles in a number of quadrilaterals, Shelly announces, "The sum of the angles of a quadrilateral is 360°."

To be effective, activities like the preceding ones need to be placed in a context. The "Phases of Learning" section presents guidelines on the sequencing and delivery of geometric activities within a level. The "Properties of the Model" section also provides teaching advice. In particular, they suggest that geometric activities should not reduce the level of the geometric content, that whenever possible, materials should set the stage for further learning, and that language is important in the development and assessment of geometric understandings. These ideas are discussed further below.

Too often, geometry is taught in a mechanical way. Consider the fact that the sum of the angles of a triangle is 180°. Frequently, this fact is established by generalizing after measuring the angles of a few triangles, or worse, students are simply told the information. The latter tactic is an example of the reduction of the content level. Level-1 activities, such as the coloring of angles in a triangular grid (fig. 1.4) and the extension of that activity to identifying parallel lines in the grid, provide the student with a powerful means, both inductively and deductively, for understanding the concept. Insight into the reason why the angle sum is 180° is obtained from the grid work and concomitantly the groundwork is laid for the formal proof at level 3. An additional bonus with this particular development is that the same structure can be reused to demonstrate that the measure of the exterior angle of a triangle equals the sum of the measures of the two interior angles.

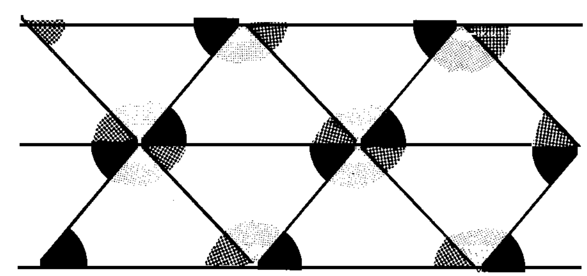
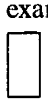


Fig. 1.4

Language, as well as thoughtfully chosen materials, plays an important role in the development of geometric thinking. It is essential that children talk about their linguistic associations for words and symbols and that they use that vocabulary. Such verbalization requires students to articulate consciously what might otherwise be vague and undeveloped ideas. It can also serve to reveal immature or misconceived ideas. Initially, children should be encouraged to express their geometric understandings in their own terms—"corner" for angle, "slanty" for the sides of a parallelogram, "straight" for parallel lines. Gradually, however, children should be introduced to standard terminology and encouraged to use it precisely. Just because children are using a word does not mean they attach the same meaning to it as their listener. For example, some children say that  is a right angle but that  is a left angle. Some say this shape () is a square but when turned 45 degrees (), it no longer is one. In each example, children have incorrectly focused on orientation as a determining characteristic. (Perhaps they were shown figures only in "standard" position.) They are interpreting the terms *right angle* and *square* to have a narrow meaning. Children who operate with notions like these are limiting their development. Through conversations, teachers can uncover misconceptions and incomplete notions as well as build on correct perceptions.

The teacher's use of language is also important. For example, in work on level 1, terms such as *all, some, always, never, sometimes* should be modeled and encouraged. Level-2 phrases include "it follows that . . ." and "if . . ., then. . . ." Level 3 would use and stress the meanings of *axiom, postulate, theorem, converse, necessary and sufficient,* and so on.

Teacher questioning is a crucial factor in directing student thinking. At all levels, asking children how they "know" is important. It is not enough, for example, for students at level 2 to be asked what is the sum of the angles of a pentagon. They should be challenged to explain why and to think about their explanation—could it be shown another way? "Raising appropriate questions, allowing a sufficient response-time and discussing the quality of the answers are methods that take into account level of thinking" (Geddes et al. 1985, p. 242).

For growth to occur, it is essential to match instruction with the student's level. Thus teachers must learn to identify students' levels of geometric thought. Because the nature of a student's geometric explanations reflects that student's level of thinking, questioning is an important assessment tool. As an example, consider responses to the questions "What type of figure is this?  How do you know?" Students at each level are able to respond "rectangle" to the first question. (If a student does not know how to name the figure, he or she is not at level 0 for rectangles.) Examples of level-specific responses to the second question are given below. In parentheses is a brief explanation of why the statement reflects the assigned level.

Level 0: "It looks like one!" or "Because it looks like a door." (The answer is based on a visual model.)

Level 1: "Four sides, closed, two long sides, two shorter sides, opposite sides parallel, four right angles . . ." (Properties are listed; redundancies are not seen.)

Level 2: "It is a parallelogram with right angles." (The student attempts to give a minimum number of properties. If queried, she would indicate that she knows it is redundant in this example to say that opposite sides are congruent.)

Level 3: "This can be proved if I know this figure is a parallelogram and that one angle is a right angle." (The student seeks to prove the fact deductively.)

Additional examples of level-specific student behaviors can be found in *An Investigation of the van Hiele Model of Thinking in Geometry among Adolescents* (Geddes et al. 1985, pp. 62–78) and in "Characterizing the van Hiele Levels of Development in Geometry" (Burger and Shaughnessy 1986, pp. 41–45).

The model of geometric thought and the phases of learning developed by the van Hieles propose a means for identifying a student's level of geometric maturity and suggest ways to help students to progress through the levels. Instruction rather than maturation is highlighted as the most significant factor contributing to this development. Research has supported the accuracy of the model for assessing student understandings of geometry (Burger 1985; Burger and Shaughnessy 1986; Geddes et al. 1982; Geddes, Fuys, and Tischler 1985; Mayberry 1981; Shaughnessy and Burger 1985; Usiskin 1982). It has also shown that materials and methodology can be designed to match levels and to promote growth through the levels (Burger 1985; Burger and Shaughnessy 1986; Geddes et al. 1982; Geddes, Fuys, and Tischler 1985; Shaughnessy and Burger 1985). The need now is for classroom teachers and researchers to refine the phases of learning, develop van Hiele–based materials, and implement those materials and philosophies in the classroom setting. Geometric thinking can be accessible to everyone.

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